PART 1: QUESTIONS

Name:________________________________ **Age:**_____ **Id:**_______________ **Course:**__________

Limits - Exam 2 **Lesson: 4-6**

Instructions:

- Please begin by printing your Name, your Age, your Student Id , and your Course Name in the box above and in the box on the solution sheet.
- You have 90 minutes (class period) for this exam.
- You can not use any calculator, computer, cellphone, or other assistance device on this exam. However, you can set our flag to ask permission to consult your own one two-sided-sheet notes at any point during the exam (You can write concepts, formulas, properties, and procedures, but questions and their solutions from books or previous exams are not allowed in your notes).
- Each multiple-choice question is worth 5 points and each extra essay-question is worth from 0 to 5 points. (Even a simple related formula can worth some points).
- Set up your flag if you have a question.
- Relax and use strategies to improve your performance.

Exam Strategies to get the best performance:

- Spend 5 minutes reading your exam. Use this time to classify each Question in (E) Easy, (M) Medium, and (D) Difficult.
- Be confident by solving the easy questions first then the medium questions.
- Be sure to check each solution. In average, you only need 30 seconds to test it. (Use good sense).
- Don't waste too much time on a question even if you know how to solve it. Instead, skip the question and put a circle around the problem number to work on it later. In average, the easy and medium questions take up half of the exam time.
- Solving the all of the easy and medium question will already guarantee a minimum grade. Now, you are much more confident and motivated to solve the difficult or skipped questions.
- Be patient and try not to leave the exam early. Use the remaining time to double check your solutions.

1. Given:

I. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

II. $\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = 1$

III. $\lim\limits_{x\to 0} \left(1+\dfrac{1}{x}\right)^{x} = e$

IV. $\lim\limits_{x\to \infty} \left(1+\dfrac{1}{x}\right)^{x} = e$

a) Only I and III are correct
b) Only I and IV are correct
c) Only II and III are correct
d) Only II and IV are correct
e) None of the above.

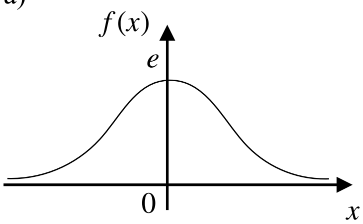
Solution: b

The two special limits are:

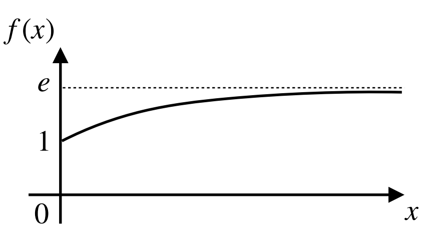
$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x\to \infty} \left(1+\frac{1}{x}\right)^{x} = e.$$

2. The graph of $f(x) = \dfrac{\sin x}{x}$ is:

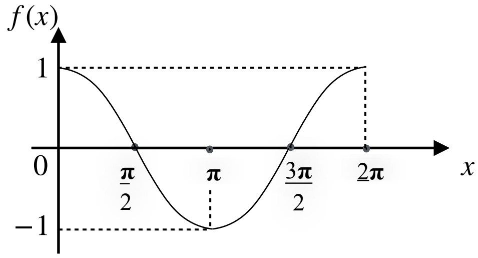
a)



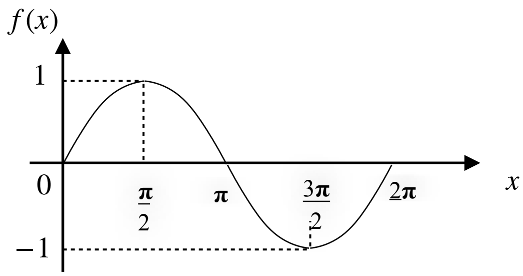
b)



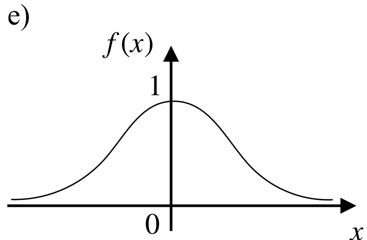
c)



d)

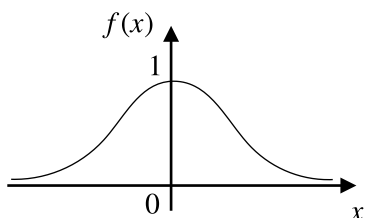


e)



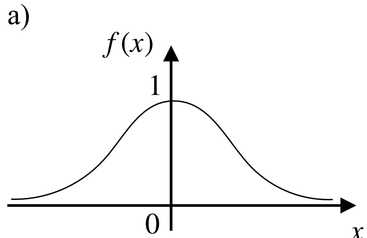
Solution: e

The graph of $f(x) = \dfrac{\sin x}{x}$ is:

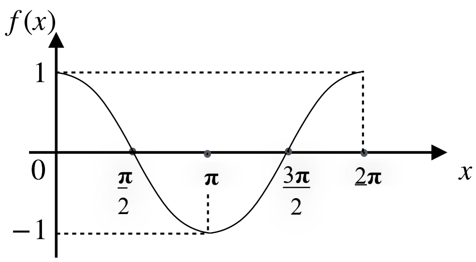


3. The graph of $f(x) = \left(1+\dfrac{1}{x}\right)^{x}$ is:

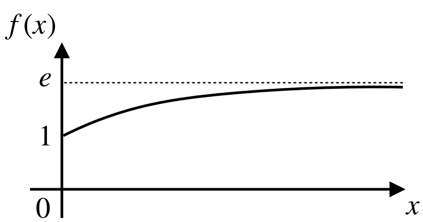
a)



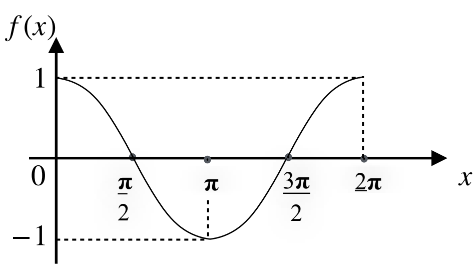
b)



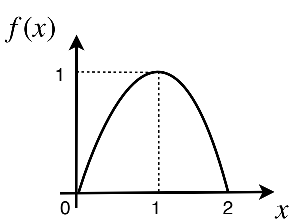
c)



d)

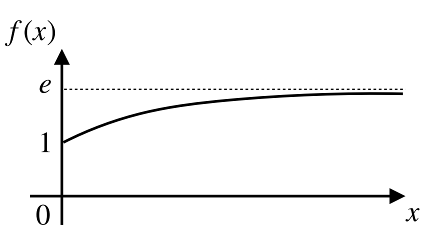


e)



Solution: c

The graph of $f(x) = \left(1 + \frac{1}{x}\right)^x$ is:



4. Given that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ then $\lim_{x \to 0} \frac{\sin x}{2x}$ is:

a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$ d) $\frac{1}{5}$ e) None of the above.

Solution: a

$$\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2}.$$

5. Given that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ then $\lim_{x \to 0} \frac{\sin 6x}{x}$ is:

a) 2 b) 3 c) 4 d) 5 e) None of the above.

Solution: e

$$\lim_{x \to 0} \frac{\sin 6x}{x} = \lim_{x \to 0} \frac{6 \sin(6x)}{(6x)} = 6 \lim_{x \to 0} \frac{\sin(6x)}{(6x)} = 6$$

6. Solve:

$$\lim_{x \to 0} \frac{(1 - \cos x)}{5x^2}$$

Hint: Multiply the numerator and the denominator per $(1 + \cos x)$.

a) $\frac{1}{2}$ b) $\frac{1}{4}$ c) $\frac{1}{6}$ d) $\frac{1}{8}$ e) None of the above.

Solution: e

$$\lim_{x \to 0} \frac{(1 - \cos x)}{5x^2} = \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{5x^2(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{(1 - \cos^2 x)}{5x^2(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{(\sin^2 x)}{5x^2(1 + \cos x)}$$

$$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 \left(\lim_{x \to 0} \frac{1}{5(1 + \cos x)}\right) = \frac{1}{10}.$$

7. Given that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$ then $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x}$ is:

a) e^2 b) e^3 c) e^4 d) e^5 e) None of the above.

Solution: d

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x} = \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^5 = e^5.$$

8. Given that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$ then $\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{x}$ is:

a) $\sqrt{e}$ b) $\sqrt[3]{e}$ c) $\sqrt[4]{e}$ d) $\sqrt[5]{e}$ e) None of the above.

Solution: b

$$\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{x} = \left[\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{\frac{3x}{3}}\right]$$

Let $y = 3x$

$$\left[\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{\frac{3x}{3}}\right] \Rightarrow \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{\frac{y}{3}}\right]$$

$$\left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{\frac{y}{3}}\right] = \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^{\frac{1}{3}} = e^{\frac{1}{3}} = \sqrt[3]{e}$$

9. Given that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$ then $\lim_{x\to\infty}\left(1+\frac{6}{x}\right)^{x}$ is:

a) e^2 b) e^3 c) e^4 d) e^5 e) None of the above.

Solution: c

$$\lim_{x\to\infty}\left(1+\frac{6}{x}\right)^{x} = \left[\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}}\right)^{6\left(\frac{x}{6}\right)}\right]$$

Let $y = \frac{x}{6}$

$$\left[\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}}\right)^{6\left(\frac{x}{6}\right)}\right] \Rightarrow \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6y}\right]$$

$$\left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6y}\right] = \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^6 = e^6.$$

10. Given $f(x) = 2x^3 + 3x + 2$.

Find the derivative $f'(x)$.

a) $f'(x) = 6x^2 + 3$
b) $f'(x) = 9x^2 - 3$
c) $f'(x) = 12x^2 - 2$
d) $f'(x) = 15x^2 - 3$
e) None of the above.

Solution: a

$f(x) = 2x^3 + 3x + 2$

Applying the Power rule $y = x^n \Rightarrow y' = n\,x^{n-1}$ in each term of the polynomial we have:

$f'(x) = (3)(2)x^{3-1} + (1)(3)x^{1-1} \Rightarrow f'(x) = 6x^2 + 3.$

11. Given $u = x^2 - 4$ and $v = x^2 + 2$.

Find the derivative of $y = uv$.

a) $y' = 4x^3 - 6x$
b) $y' = 4x^3 + 6x$
c) $y' = 4x^3 + 2x$
d) $y' = 4x^3 - 4x$
e) None of the above.

Solution: d

$u = x^2 - 4$ and $v = x^2 + 2$.
Applying the Product rule we have:
$y = uv \Rightarrow y' = u'v + uv'.$
$y = (x^2-4)(x^2+2) \Rightarrow y' = (2x)(x^2+2) + (x^2-4)(2x)$
$y' = 2x^3 + 4x + 2x^3 - 8x \Rightarrow y' = 4x^3 - 4x.$

12. Given $y = \sin x - 3x^2$.

Find the derivative y'. Hint: $y = \sin x \Rightarrow y' = \cos x$.

a) $y' = \cos x + 2$
b) $y' = \cos x + 8x$
c) $y' = \cos x - 6x$
d) $y' = \cos x + 3x^2$
e) None of the above.

Solution: c

$y = \sin x - 3x^2 \Rightarrow y' = \cos x - 6x$.

13. Given:

I. L'Hopital's rule uses derivatives to help evaluate limits involving indeterminate fractions such: $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

II. L'Hopital's rule should be used to check limits problems, or solve exam questions with permission from your professor.

III. L'Hopital's rule is a strong tool to solve difficult limits.

a) Only I and II are correct.
b) Only I and III are correct.
c) Only II and III are correct.
d) All alternatives I, II and III are correct.
e) None of the above.

Solution: d

All alternatives are correct.

14. In short, L'Hopital's rule is:

a) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pi}{e}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \Rightarrow$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{g'(x)}{f'(x)}$.

b) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{\infty}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \Rightarrow$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

c) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{e}{\pi}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \Rightarrow$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{g'(x)}{f'(x)}$.

d) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \Rightarrow$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

e) None of the above.

Solution: d

L'Hopital's rule is:

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \Rightarrow$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

15. Solve: $\lim_{x\to 4} \frac{x^3 - 16x}{x - 4}$

a) 2
b) 8
c) 18
d) 32
e) None of the above.

Solution: c

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ (L'Hopital)

$$\lim_{x\to 4}\frac{x^3-16x}{x-4}=\frac{o}{o}\Rightarrow\lim_{x\to 4}\frac{3x^2-16}{1}=32.$$

16. Solve: $\lim_{x\to 1}\frac{x^6-x}{x-1}$

a) 2
b) 3
c) 4
d) 5
e) None of the above.

Solution: d

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}\Rightarrow\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\quad\text{(L'Hopital)}$$

$$\lim_{x\to 1}\frac{x^6-x}{x-1}=\frac{o}{o}\Rightarrow\lim_{x\to 1}\frac{6x^5-1}{1}=5.$$

17. Given:

I. L' Hopital's rule can be extended to solve indeterminate powers such as: $1^\infty, 0^0$, or ∞^0.

II. Let $f(x)$ and $g(x)$ be functions on x such that $f(x)>0$ in a neighborhood of a real number "a".

Then:

$$\lim_{x\to a}f(x)^{g(x)}=e^{\lim_{x\to a}\left[g(x)\log_e f(x)\right]}$$

III. The indeterminate power limit $\lim_{x\to a}f(x)^{g(x)}$ can be rewritten as an indeterminate fraction, which can be easily solved with L'Hopital's rule.

Hint: Use I, II, and III to solve some questions in your exam.

a) Only I and II are correct.
b) Only I and III are correct.
c) Only II and III are correct.
d) All alternatives I, II and III are correct.
e) None of the above.

Solution: d

All alternatives are correct.

18. Solve:

$$\lim_{x\to\infty}x^{\left[\frac{5}{x}\right]}$$

a) 0
b) e
c) e^2
d) e^3
e) None of the above.

Solution: e

Let $f(x)=x$ and $g(x)=\frac{5}{x}$.

$$\lim_{x\to a}f(x)^{g(x)}=\lim_{x\to a}e^{\left[g(x)\log_e f(x)\right]}$$

$$\lim_{x\to\infty}x^{\left[\frac{5}{x}\right]}=\lim_{x\to\infty}e^{\left[\frac{5\log_e x}{x}\right]}=e^{\lim_{x\to\infty}\left[\frac{5\log_e x}{x}\right]}$$

$$\lim_{x\to\infty}\left[\frac{5\log_e x}{x}\right]=\frac{\infty}{\infty}\text{, then by L'Hopital's rule:}$$

$$\lim_{x\to\infty}\left[\frac{5\log_e x}{x}\right]=\lim_{x\to\infty}\frac{\frac{5}{x}}{1}=\lim_{x\to\infty}\frac{5}{x}=0.$$

Thus, $\lim_{x\to\infty}x^{\left[\frac{5}{x}\right]}=e^{\lim_{x\to\infty}\left[\frac{5\log_e x}{x}\right]}=e^0=1.$

19. Solve:

$$\lim_{x\to 1}(x+e)^0$$

a) 4
b) 3
c) 2
d) 1
e) None of the above.

Solution: d

$\lim_{x \to 1} (x + e)^0 = \lim_{x \to 1} 1 = 1.$

20. Solve: $\lim_{x \to 1} \frac{x^5 - 1}{x^6 - 1}$

a) $\frac{3}{4}$ b) $\frac{4}{5}$ c) $\frac{5}{6}$ d) $\frac{6}{7}$ e) None of the above.

Solution: c

$\lim_{x \to 1} \frac{x^5 - 1}{x^6 - 1} = \frac{0}{0}$, then by L'Hopital's rule:

$\lim_{x \to 1} \frac{x^5 - 1}{x^6 - 1} = \lim_{x \to 1} \frac{5x^4}{6x^5} = \frac{5}{6}.$

PART 2: SOLUTIONS

Consulting

Name:______________________________ **Age:**_____ **Id:**______________ **Course:**__________

Multiple-Choice Answers

Questions	A	B	C	D	E
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					

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	Points	Max
Multiple Choice		100
Extra Points		25
Consulting		10
Age Points		25
Total Performance		160
Grade		A

Extra Questions

21. Prove that $\lim_{x\to 0} \dfrac{5 \sin x}{x} = 5.$

Hint: Use L'Hopital's rule.

Solution:

$$\lim_{x\to 0} \frac{5 \sin x}{x} = \frac{0}{0} \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ (L'Hopital)}$$

$$\lim_{x\to 0} \frac{5 \sin x}{x} = \lim_{x\to 0} \frac{5 \cos x}{1} = 5.$$

22. Solve: $\lim_{x\to 0} \dfrac{\sin 4x}{5x}$

Solution:

$$\lim_{x\to 0} \frac{\sin 4x}{5x} = \lim_{x\to 0} \frac{\frac{\sin(4x)}{(4x)}}{\frac{(5x)}{(4x)}} = \frac{4}{5} \lim_{x\to 0} \frac{\sin(4x)}{(4x)}$$

$y = 4x$. Note: $x \to 0 \Rightarrow y \to 0$. Then:

$$\frac{4}{5} \lim_{x\to 0} \frac{\sin(4x)}{(4x)} = \frac{4}{5} \lim_{y\to 0} \frac{\sin(y)}{(y)} = \frac{4}{5}.$$

23. Solve:

$$\lim_{x\to\infty}\left(1+\frac{4}{5x}\right)^x$$

Solution:

$$\lim_{x\to\infty}\left(1+\frac{4}{5x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{5x}{4}}\right)^x$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\left(\frac{5x}{4}\right)}\right)^{\left(\frac{5x}{4}\right)\frac{4}{5}}$$

Let $y = \dfrac{5x}{4}$. Note: $x \to \infty \Rightarrow y \to \infty$.

$$\lim_{x\to\infty}\left(1+\frac{1}{\left(\frac{5x}{4}\right)}\right)^{\left(\frac{5x}{4}\right)\frac{4}{5}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y\frac{4}{5}}$$

$$= \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y\right]^{\frac{4}{5}} = e^{\frac{4}{5}}.$$

24. Given $u = 2x - 3$ and $v = 3x + 2$. Find the derivative of $y = uv$ using two different methods.

a) Product rule: $y = uv \Rightarrow y' = u'v + uv'$.

b) Distribution: First, multiply u and v, and after derive the polynomial.

Solution:

Let $u = 2x - 3$ and $v = 3x + 2$.

a) Product rule:

$y = uv \Rightarrow y' = u'v + uv'$.
$y = (2x-3)(3x+2)$
$y' = (2)(3x+2) + (2x-3)(3)$
$y' = 6x + 4 + 6x - 9$
$y' = 12x - 5$.

b) Distribution:

$y = (2x-3)(3x+2)$
$y = 6x^2 + 4x - 9x - 6$
$y = 6x^2 - 5x - 4$
$y' = (2)(6)x - (1)(5)$
$y' = 12x - 5$.

25. Solve: $\displaystyle\lim_{x\to 1}\frac{x^3-4x^2+5x-2}{x^3-8x^2+13x-6}$

Hint: Apply two times L'Hopital's rule.

Solution:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ (L'Hopital)}$$

Since $\displaystyle\lim_{x\to 1}\frac{x^3-4x^2+5x-2}{x^3-8x^2+13x-6} = \frac{o}{o}$ then:

$$\lim_{x\to 1}\frac{3x^2-8x+5}{3x^2-16x+13} = \lim_{x\to 1}\frac{6x-8}{6x-16} = \frac{3}{10}.$$